ON THE LAPLACE TRANSFORM OF THE STATE TRANSITION MATRIX

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Abstract. This note presents the explicit form of the Laplace transform of the state transition matrix in the case of a system matrix being a companion matrix. Starting from this special case, we show how to compute the Laplace transform of the state transition matrix for any system matrix.¹

Key words. Laplace transform, state transition matrix, resolvant matrix, matrix exponential, companion matrix, Frobenius normal form

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1. Introduction. For the calculation of the trajectory of a linear system \( \dot{x} = Ax + Bu, \) the state transition matrix, \( e^{At} \), is frequently used. This matrix exponential, \( e^{At} \), can be computed in a number of ways [2, 3, 4]. One method uses the Laplace transform

\[ \mathcal{L}\{e^{At}\} = (sI - A)^{-1}. \]  

This Laplace transform of the state transition matrix also appears in the transfer function,

\[ G(s) = C(sI - A)^{-1}B, \]  

of a linear system \( \dot{x} = Ax + Bu, y = Cx. \) Thus, it is desirable to obtain the explicit form of the so-called resolvant matrix \((sI - A)^{-1}\) without a symbolic matrix inversion of the parameter dependent matrix \( sI - A. \)

One method is provided by the Leverrie-Souriau-Faddeeva-Frame method [5], which results in the following formulas:

\[ (sI - A)^{-1} = \frac{1}{N(s)} \sum_{k=0}^{n-1} \left( \sum_{l=0}^{k} a_{n-k+l}s^l \right) A^{n-1-k}, \]  

\[ (sI - A)^{-1} = \frac{1}{N(s)} \sum_{k=0}^{n-1} \left( \sum_{l=0}^{k} a_{n-k+l}A^l \right) s^{n-1-k}, \]  

where

\[ N(s) = det(sI - A)^{-1} = \sum_{k=0}^{n} a_k s^k \]  

is the characteristic polynomial of \( A. \)

Another method is introduced in this article.

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¹This note contains all results of the German article [1] and provides some additional comments and illustrative examples.
2. The special case of companion matrices. If a system matrix, $A$, is a companion matrix, it is possible to explicitly state the Laplace transformed state transition matrix for this system matrix. The details are given by

**Theorem 2.1.** For a given $n \times n$ companion matrix

$$A_c = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix}$$ (2.1)

we have

$$(sI - A_c)^{-1} = \frac{1}{N(s)} \mathbf{H}(s),$$ (2.2)

where $N(s) = \sum_{i=0}^{n} a_i s^i$ is the characteristic polynomial of $A_c$ with $a_n = 1$ and where

$$h_{t,m} = \begin{cases} -\sum_{i=0}^{t-1} a_i s^{i+m-l-1} & \text{for } l < m, \\ +\sum_{i=t}^{n} a_i s^{i+m-l-1} & \text{for } l \geq m. \end{cases}$$ (2.3)

are the elements of the matrix $\mathbf{H}$.

**Proof.** The elements of $A_c$ given by Eq.(2.1) can be expressed as $a_{k,l} = \delta_{k-1,l} - a_{k-1} \delta_{n,l}$, using the Kronecker delta

$$\delta_{i,j} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$ (2.4)

We define $F = sI - A_c$. Then we have

$$f_{k,l} = s\delta_{k,l} - \delta_{k-1,l} + a_{k-1} \delta_{n,l}.$$ (2.5)

The elements of the matrix $\mathbf{H}$ are defined by Eq.(2.3).

Instead of proving the assertion $F^{-1} = (sI - A_c)^{-1} = \frac{1}{N(s)} \mathbf{H}$ given in Eq.(2.2), we prove the equivalent assertion $FH = N(s) \cdot I$ as follows:

The elements of the matrix $\mathbf{FH}$ are given by

$$(\mathbf{FH})_{k,m} = \sum_{l=1}^{n} f_{k,l} h_{l,m},$$ (2.6)

$$= \sum_{l=1}^{n} (s\delta_{k,l}h_{l,m} - \delta_{k-1,l} h_{l,m} + a_{k-1} \delta_{n,l} h_{l,m}).$$

Due to the appearance of the $\delta$-terms in Eq.(2.6), the three products inside the brackets can be non-zero iff the index $l$ satisfies $l = k$, $l = k - 1$, and $l = n$, respectively. Consequently, the summation over the index $l$ is easy to perform and we obtain

$$(\mathbf{FH})_{k,m} = sh_{k,m} - h_{k-1,m} + a_{k-1} h_{n,m},$$

where we extend the definition of $h_{l,m}$ by setting $h_{0,m} = 0$. According to the definition of the matrix $\mathbf{H}$ given by Eq.(2.3), the elements $h_{n,m}$ are

$$h_{n,m} = \sum_{i=n}^{n} a_i s^{i+m-n-1} = a_n s^{m-1} = s^{m-1}$$ (2.7)
and thus

\[(\text{FH})_{k,m} = sh_{k,m} - h_{k-1,m} + a_{k-1}s^{m-1}.\]  \hspace{1cm} (2.8)

In the following, we will substitute the elements of \(H\) in Eq.(2.8) by using their according definition given by Eq.(2.3). We consider three different cases:

Case 1: \(k = m\) (elements on the main diagonal of \(\text{FH}\))

\[(\text{FH})_{k,k} = sh_{k,k} - h_{k-1,k} + a_{k-1}s^{k-1}\]

\[= s \sum_{i=k}^{n} a_i s^{i-1} + \sum_{i=0}^{k-2} a_i s^i + a_{k-1}s^{k-1}\]

\[= n \sum_{i=0}^{k} a_i s^i = N(s).\]  \hspace{1cm} (2.9)

Case 2: \(k < m\) (elements in the upper right part of \(\text{FH}\))

\[(\text{FH})_{k,m} = -s \sum_{i=0}^{k-1} a_i s^{i+m-k-1} + \sum_{i=0}^{k-2} a_i s^i + a_{k-1}s^{m-1}\]

\[= -a_{k-1}s^{m-1} + a_{k-1}s^{m-1} = 0.\]  \hspace{1cm} (2.10)

Case 3: \(k > m\) (elements in the lower left part of \(\text{FH}\))

\[(\text{FH})_{k,m} = s \sum_{i=k}^{n} a_i s^{i+m-k-1} - \sum_{i=k-1}^{n} a_i s^i + a_{k-1}s^{m-1}\]

\[= -a_{k-1}s^{m-1} + a_{k-1}s^{m-1} = 0.\]  \hspace{1cm} (2.11)

Combining the results of the three cases above, we obtain \((\text{FH})_{k,m} = N(s) \cdot \delta_{k,m}\), i.e., \(\text{FH} = N(s) \cdot \text{I}\).

In the following sections, we will generalize the results of Theorem 2.1 for the case of arbitrary square matrices \(A\).

3. The prevalent case. First, we consider the case that the matrix \(A\) can be transformed into a companion matrix \(A_c = T^{-1}AT\) by a coordinate transformation with a regular matrix \(T\) \([6, 7]\. This transformation is possible iff the matrix \(A\) is non-derogatory\(^\text{2}\). With the inverse transformation, \(A = TA_cT^{-1}\), we obtain

\[(sI - A)^{-1} = (sI - TA_cT^{-1})^{-1} = T(sI - A_c)^{-1}T^{-1}\]  \hspace{1cm} (3.1)

and, after inserting Eq.(2.2) of Theorem 2.1,

\[(sI - A)^{-1} = \frac{1}{N(s)}TH(s)T^{-1}.\] \hspace{1cm} (3.2)

Since the matrix \(H(s)\) is polynomial in \(s\), we can express Eq.(3.2) as

\[(sI - A)^{-1} = \frac{1}{N(s)} \sum_{i=0}^{n-1} TH_i T^{-1} s^i\] \hspace{1cm} (3.3)

\(^\text{2}\)A square matrix is non-derogatory iff the eigenspace of each eigenvalue is one-dimensional.
with constant matrices $H_i$. The benefit of Eq.(3.3) is that the main part in the evaluation of the right hand side does no longer require the symbolic inversion of the parameter dependent matrix $sI - A$. Thus, we can calculate $(sI - A)^{-1}$ using only numerical operations, e.g., by employing a software package like MATLAB.

As an application, let us now consider observable SISO-systems $\dot{x} = Ax + bu$, which always have non-derogatory system matrices $A$. If such a system is given in observable canonical form, then the system matrix, $A$, is a companion matrix, $A_c$, as given by Eq.(2.1) \cite{8}. Thus, Theorem 2.1 is directly applicable and $(sI - A)^{-1}$ is given by (2.2).

Otherwise, an observable SISO-system can be transformed into observable canonical form and, thus, $(sI - A)^{-1}$ is given by Eq.(3.2). The transformation matrix $T$ can be calculated easily in the following well known manner \cite{8}:

$$T = \begin{bmatrix} \cdots \vdots \vdots \vdots \end{bmatrix}, \quad M = \begin{bmatrix} c^T \\ c^TA \\ \vdots \\ c^T A^{n-1} \end{bmatrix} \tag{3.4}$$

If a controllable SISO-system is given in controllable canonical form, then the system matrix, $A$, is the transpose of the companion matrix, $A_c$, in Eq.(2.1) \cite{8}. Then, we obtain

$$(sI - A)^{-1} = (sI - A^T)^{-1} = \frac{1}{N(s)} H^T(s) \quad \text{(3.5)}$$

by simply transposing Eq.(2.2). Otherwise, a controllable SISO-system can be transformed into controllable canonical form with a transformation matrix, $T$, \cite{8} similar to Eq.(3.4) and we obtain

$$(sI - A)^{-1} = \frac{1}{N(s)} TH^T(s)T^{-1} \quad \text{(3.6)}$$

As an illustrative example, we consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & -1.2 \\ 1 & 0 & 0 & -0.4 \\ 0 & 1 & 0 & -2.3 \\ 0 & 0 & 1 & -1.7 \end{bmatrix}$$

Using Eq.(2.3) in Theorem 2.1, we directly obtain

$$\left( sI - A_c \right)^{-1} = \frac{1}{N(s)} H(s) = \frac{1}{s^4 + 1.7s^3 + 2.3s^2 + 0.4s + 1.2}$$

$$= \begin{bmatrix} 0.4 + 2.3s + 1.7s^2 + s^3 & -1.2 & -1.2s & -1.2s^2 \\ 2.3 + 1.7s + s^2 & 2.3s + 1.7s^2 + s^3 & -1.2 - 0.4s & -1.2s - 0.4s^2 \\ 1.7 + s & 1.7s + s^2 & 1.7s^2 + s^3 & -1.2 - 0.4s - 2.3s^2 \end{bmatrix}$$

\footnote{In an alternatively used definition of the observable canonical form \cite{9}, the ordering of both the lines and the columns of the system matrix is reversed. In this case, also the ordering of the lines and columns of the matrix $H$ have to be reversed in Eq.(2.2) and Eq.(3.2), respectively.}
4. The general case. Now, we consider the general case, which also includes derogatory matrices, \( \mathbf{A} \). Any matrix, \( \mathbf{A} \), can be transformed by a coordinate transformation with a regular matrix, \( \mathbf{T} \), into Frobenius normal form [6, 7]. The transformed matrix, \( \mathbf{A}_F = \mathbf{T}^{-1} \mathbf{A} \mathbf{T} = \text{diag}(\mathbf{A}_c^{(1)}, ..., \mathbf{A}_c^{(k)}) \), is a block diagonal matrix, where the block matrices \( \mathbf{A}_c^{(i)} \) are companion matrices. For each block matrix \( \mathbf{A}_c^{(i)} \), we can obtain the matrices

\[
\mathbf{N}^{(i)}(s) \mathbf{H}^{(i)}(s) = (s \mathbf{I} - \mathbf{A}_c^{(i)})^{-1},
\]

(4.1)

which is equivalent to

\[
\mathbf{M}(s) = \text{diag} \left( \frac{1}{N^{(1)}(s)} \mathbf{H}^{(1)}(s), ..., \frac{1}{N^{(k)}(s)} \mathbf{H}^{(k)}(s) \right),
\]

(4.2)

With Eq.(4.2) and

\[
(s \mathbf{I} - \mathbf{A})^{-1} = (s \mathbf{I} - \mathbf{T} \mathbf{A}_F \mathbf{T}^{-1})^{-1} = \mathbf{T}(s \mathbf{I} - \mathbf{A}_F)^{-1} \mathbf{T}^{-1},
\]

(4.3)

we obtain, in the general case, for the Laplace transform of the state transition matrix:

\[
(s \mathbf{I} - \mathbf{A})^{-1} = \mathbf{T} \mathbf{M}(s) \mathbf{T}^{-1},
\]

(4.4)

where \( \mathbf{M}(s) \) is given by Eq.(4.1). The transformation matrix \( \mathbf{T} \) can be computed numerically by an efficient algorithm [7] in general. Similar to the matrix \( \mathbf{H}(s) \) in Eq.(3.3), the matrix \( \mathbf{M}(s) \) is polynomial in \( s \) and, thus, \( (s \mathbf{I} - \mathbf{A})^{-1} \) can be determined without symbolic calculation once again.

5. Conclusion. In this note, we showed that the Laplace transform of the state transition matrix can be stated explicitly if the system matrix is a companion matrix and can be determined without symbolic matrix inversion in the general case of arbitrary system matrices.

REFERENCES